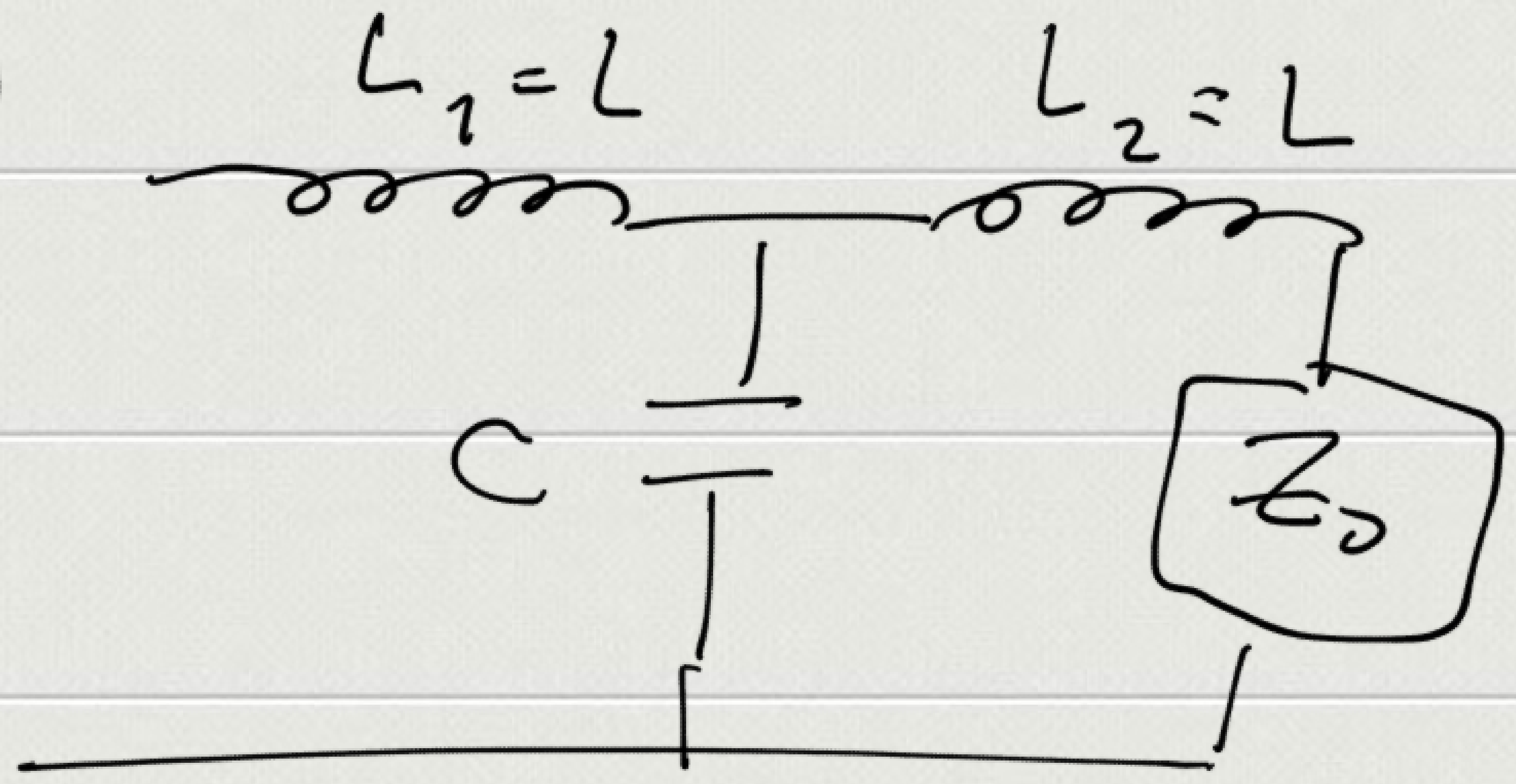
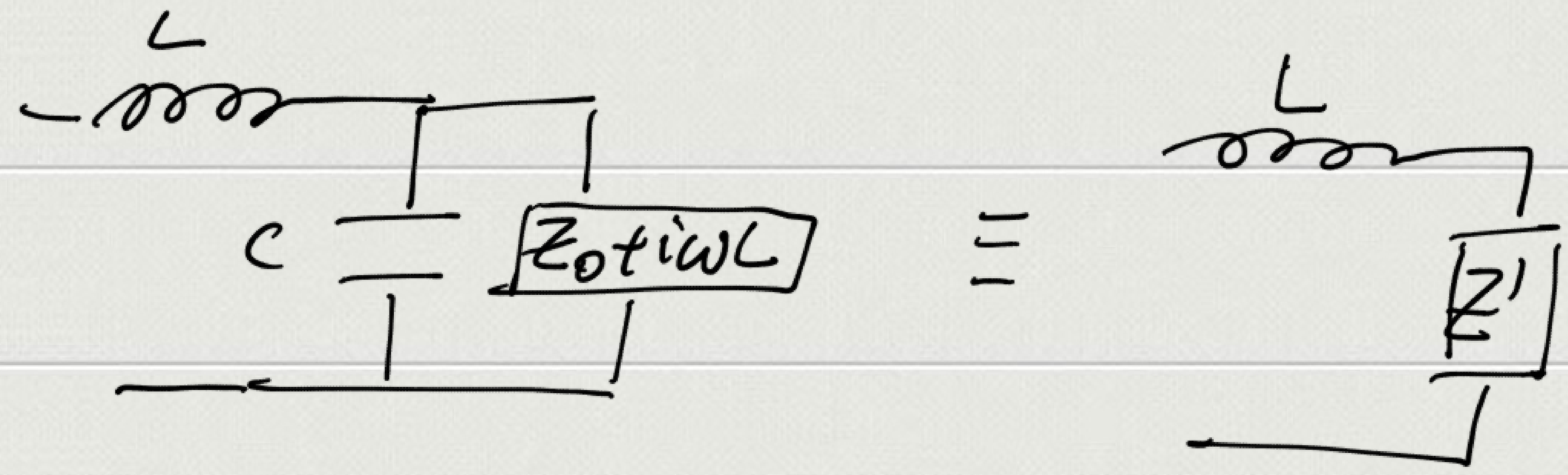
HOMEWORK 9

8.34



L_2 and Z_0 are in series. Their total impedance is $Z_0 + i\omega L$. Then this new impedance is in parallel with C



with $\frac{1}{Z'} = \frac{1}{Z_0 + i\omega L} + i\omega C = \frac{1 + i\omega C Z_0 - \omega^2 LC}{Z_0 + i\omega L}$

$$Z' = \frac{Z_0 + i\omega L}{1 + i\omega C Z_0 - \omega^2 LC}$$

Then, Z' is in series with L. So the total impedance is $Z = Z' + i\omega L$

$$Z = \frac{Z_0 + i\omega L}{1 - \omega^2 LC + i\omega C Z_0} + i\omega L$$

We want $Z = Z_0$, therefore

$$\frac{Z_0 + i\omega L}{1 - \omega^2 LC + i\omega C Z_0} + i\omega L = Z_0$$

$$Z_0 + i\omega L + i\omega L - i\omega^3 L^2 C - \omega^2 LC Z_0 = Z_0 - \omega^2 LC Z_0 + i\omega C Z_0^2$$

$$2i\omega L - i\omega^3 L^2 C = i\omega C Z_0^2$$

$$2L - \omega^2 L^2 C = C Z_0^2$$

$$\boxed{Z_0 = \sqrt{\frac{2L - \omega^2 L^2 C}{C}}}$$

This is a pure resistance if the argument of the square root is positive, ie,

$$2L - \omega^2 L^2 C > 0 \qquad 2 > \omega^2 LC$$

$$\boxed{\omega^2 < \frac{2}{LC}} \qquad \text{At } \omega^2 = \frac{2}{LC}, \quad Z_0 = 0$$

8.38 We have R and C in series, and then
(a) the combination in parallel with R

R & C in series: $Z' = R + \frac{1}{i\omega C} = \frac{1 + i\omega CR}{i\omega C}$

But $\omega RC = 1$ and $\omega C = \frac{1}{R} \Rightarrow Z' = R\frac{(1+i)}{i} = R(1-i)$

Then, Z' in parallel with R gives

$$\frac{1}{Z} = \frac{1}{Z'} + \frac{1}{R} = \frac{1}{R}\left(\frac{1}{1-i} + 1\right) = \frac{1}{R}\,\frac{2-i}{1-i}$$

$$\Rightarrow \boxed{Z = R\,\frac{1-i}{2-i}} \quad \text{or} \quad Z = R\,\frac{(1-i)(2+i)}{5}$$

$$\boxed{Z = \frac{R}{5}(3-i)}$$

(b) $\mathcal{E} = \mathcal{E}_0 e^{i\omega t}$

$$I = \frac{\mathcal{E}}{Z} = \frac{\mathcal{E}_0 e^{i\omega t}}{R\left(\frac{1}{5}\right)(3-i)} = 5\frac{\mathcal{E}_0}{R} e^{i\omega t} \frac{1}{3-i}$$

Let's write $\frac{1}{3-i}$ as $Ae^{i\phi}$

$$\frac{1}{3-i} = \frac{3+i}{10} \qquad A^2 = \sqrt{\left(\frac{3}{10}\right)^2 + \left(\frac{1}{10}\right)^2} = \frac{1}{\sqrt{10}}$$

$$\boxed{\phi = \tan^{-1} \tfrac{1}{3}}$$

Then $I = 5\frac{\mathcal{E}_0}{R}\frac{1}{\sqrt{10}} e^{i(\omega t + \phi)} = \frac{5\sqrt{10}}{10}\frac{\mathcal{E}_0}{R} e^{i(\omega t + \phi)}$

writing $I = I_0 \cos(\omega t + \phi) = I_0 \operatorname{Re}\left[e^{i(\omega t + \phi)}\right]$

we get $\boxed{I_0 = \frac{5\sqrt{10}}{10}\frac{\mathcal{E}_0}{R} = \frac{\sqrt{10}}{2}\frac{\mathcal{E}_0}{R}}$

(c) Average power $\langle P \rangle = I_0 \mathcal{E}_0 \frac{\cos\phi}{2}$

$\langle P \rangle = \frac{\sqrt{10}}{4}\frac{\mathcal{E}_0^2}{R} \cos\phi$ _ We know $\tan\phi = 1/3$

$$\frac{\sin\phi}{\cos\phi} = \frac{1}{3} \qquad \sin\phi = \frac{1}{3}\cos\phi \qquad \sin^2\phi = \frac{1}{9}\cos^2\phi$$

$$1 - \cos^2\phi = \frac{1}{9}\cos\phi^2 \qquad \cos^2\phi = \frac{9}{10} \qquad \cos\phi = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \langle P \rangle = \frac{\sqrt{10}}{4}\frac{\mathcal{E}_0^2}{R}\frac{3}{\sqrt{10}} \qquad \boxed{\langle P \rangle = \frac{3\mathcal{E}_0^2}{4R}}$$

9.15 $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\pi b^2 \varepsilon_0}$

$$\frac{\partial E}{\partial t} = \frac{1}{\pi b^2 \varepsilon_0}\frac{\partial Q}{\partial t} = \frac{1}{\pi b^2 \varepsilon_0} I \longleftarrow$$ Field inside capacitor

Using the path suggested in the problem

$$\int_C \vec{B}\,d\vec{s} = \int_S \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\,d\vec{A}$$

$$B(r)\,2\pi r = \mu_0 \varepsilon_0 \frac{1}{\pi b^2 \varepsilon_0} I \pi r^2$$

$$\boxed{B(r) = \frac{\mu_0 I r}{2\pi b^2}}$$

9.18 Let $k = 2\pi/\lambda$

$$\vec{E} = E_0(\hat{x} + \hat{y})\sin k(z + ct)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{and} \quad \text{curl}\,\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_0 \sin k(z+ct) & E_0 \sin k(z+ct) & 0 \end{vmatrix} =$$

$$\vec{\nabla}\times\vec{E} = +E_0 k \cos k[z+ct]\,(-\hat{x}+\hat{y})$$

and this is $= -\dfrac{\partial \vec{B}}{\partial t}$

Therefore $\vec{B} = \dfrac{E_0}{c}\sin k[z+ct]\,(\hat{x}-\hat{y})$

Note $\vec{E}\perp\vec{B}$!!

9.25 $\quad u = \varepsilon_0 \langle E^2 \rangle = 4\;10^{-14}\,\dfrac{J}{m^3}$

Since $\varepsilon_0 = 8.85\;10^{-12}\,\dfrac{s^2C^2}{Kg\,m^3}$

$\langle E^2 \rangle = 4.5\;10^{-3}\,\dfrac{V^2}{m^2}$ $\qquad E_{rms} = \sqrt{\langle E^2\rangle} = 0.07\,V/m$

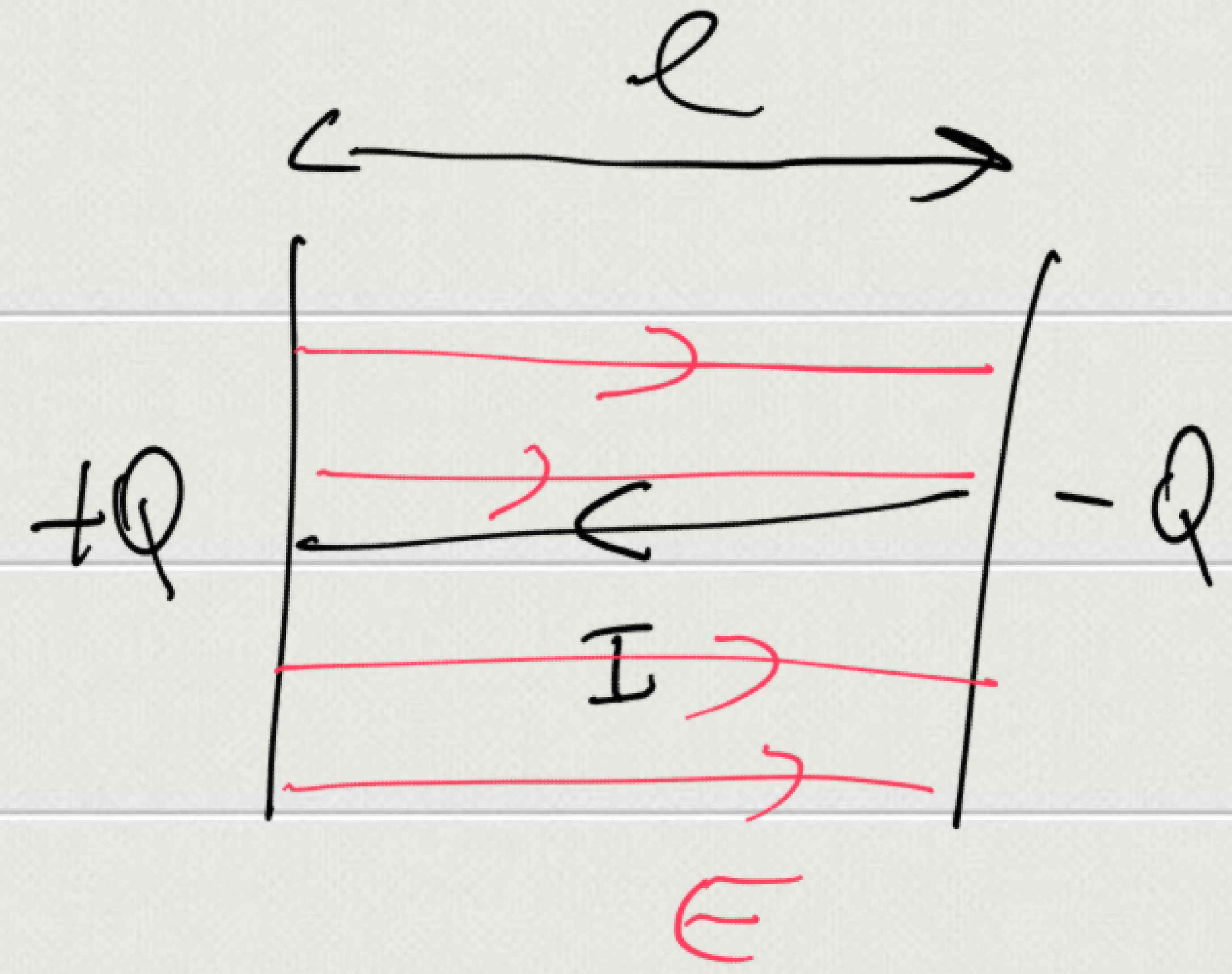
Assume that the kW radiated by the radio transmitter is radiated uniformly in space (ie, in a spherically symmetric manner) – $P = 10^3\,W$

Power density $= \dfrac{P}{4\pi R^2}$ where R = distance from transmitter

We want $\dfrac{1}{c}\,\dfrac{P}{4\pi R^2} = u = 4\cdot 10^{-14}\,J/m^3$

$$R = \sqrt{\frac{P}{4\pi c u}} = \sqrt{\frac{10^3}{4\pi\;3\;10^8\;4\;10^{-14}}} = \underline{\underline{2580\,m}}$$

9.29



The electric field lines go from one side of the capacitor to the other. Let A = area of capacitor Q = charge on capacitor $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$

Also $\frac{dQ}{dt} = I \Rightarrow I = A\varepsilon_0 \frac{\partial E}{\partial t}$

Now take a circular path inside the capacitor. Radius r

$$\oint \vec{B}\, d\vec{l} = 2\pi r\, B(r)$$

But

$$\oint \vec{B}\, d\vec{l} = \mu_0 I - \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} (\pi r^2)$$

(The minus sign is there because I and $\vec{E}$ point in opposite directions, as shown in figure)

$$\Rightarrow B(r) = \frac{\mu_0 I}{2\pi r} - \frac{\varepsilon_0 \mu_0 r}{2} = \frac{\varepsilon_0 \mu_0}{2} \frac{\partial E}{\partial t} \left(\frac{A}{\pi r} - r \right)$$

The poynting vector is $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$

$\vec{S}$ points away from wire.

The magnitude of S, at a given r is

$$S = \frac{\varepsilon_0}{2} E \frac{dE}{dt} \left(\frac{A}{\pi r} - r \right)$$

The flux of S through a cylinder of radius r and length l is

$$\text{Flux} = S(2\pi r l) = \varepsilon_0 E \frac{dE}{dt} l \left(A - \pi r^2 \right)$$

$$\text{Flux} = \frac{d}{dt} \left[\frac{\varepsilon_0}{2} E^2 l \left(A - \pi r^2 \right) \right]$$

The energy stored in the cylinder is

$$U_{\text{cylinder}} = \frac{\varepsilon_0}{2} E^2 l \pi r^2$$

The energy stored in the full capacitor is

$$U_C = \frac{\varepsilon_0}{2} E^2 l A$$

$\Rightarrow \frac{\varepsilon_0}{2} E^2 l \left(A - \pi r^2 \right)$ is the energy stored in the capacitor outside cylinder and its rate of change = flux of pointing vector.