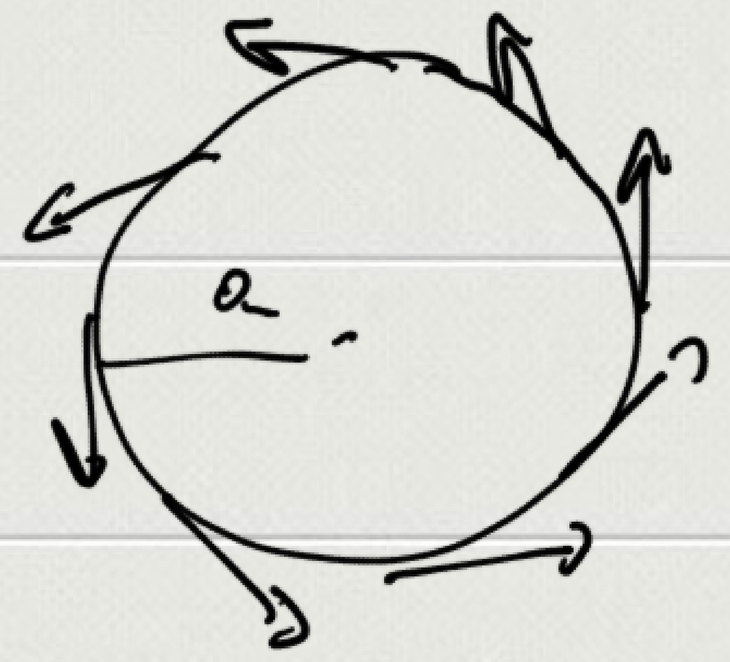
Homework 8

7.33 At any given time the flux is $\boxed{\Phi = \pi a^2 B}$.
If B changes, then Φ changes, therefore there is an emf. In magnitude $\boxed{\text{emf} = \frac{d\Phi}{dt}}$
If the ring was a conductor, we would get current flowing. But the ring is an insulator, so current does not flow. However, there is an $\vec{E}$ field associated with the emf inside the ring. Since the ring is charged, this $\vec{E}$ field will cause a force, and this force will cause a torque. Why a torque? Because the forces on each charge element are tangential:



I will not bother keeping track of signs.

$\tau = Fa = qEa$. Now we need to find E.

$\text{emf} = \frac{d\Phi}{dt}$ but $\text{emf} = \int \vec{E}\, d\vec{l} = 2\pi a E$

$$2\pi a E(t) = \pi a^2 \frac{dB}{dt} \qquad E(t) = \frac{1}{2} a \frac{dB}{dt}$$

$$\tau(t) = \frac{1}{2} q a \frac{dB}{dt}$$

Let L = angular momentum $L = I\omega$, where I is the moment of inertia.

We have $\tau = \frac{dL}{dt} = I\frac{d\omega}{dt}$

$$\frac{1}{2}qa^2\frac{dB}{dt} = I\frac{d\omega}{dt} = ma^2\frac{d\omega}{dt} \qquad (I = ma^2)$$

$\frac{d\omega}{dt} = \frac{q}{2m}\frac{dB}{dt}$ Integrate ω from $0 \to \omega$
Integrate B from $B_0 \to 0$

Up to a sign, I then get $\boxed{\omega = \frac{qB_0}{2m}}$

7.36 (a) If I_2 increases, there will be extra flux through the top circuit pointing up. Therefore the induced emf in the top circuit must make an extra downwards flux. The induced current must then be negative $\Rightarrow \mathcal{E}_1 = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt}$ (1)

Exact same argument leads to

$$\mathcal{E}_2 = -L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt} \quad (2)$$

If we were to switch the convention for +ve I_2 and +ve $\mathcal{E}_2$, we would get the opposite sign, ie, $+M\,dI/dt$

Figure (b)

(b) Key idea here is that in this case $I_1 = I_2 = I$
Also the total emf $\mathcal{E}$ is $\mathcal{E}_1 + \mathcal{E}_2$
Then adding equations (1) and (2) above, I get

$$\mathcal{E} = -(L_1 + L_2 + 2M)\frac{dI}{dt} \Rightarrow \boxed{L' = L_1 + L_2 + 2M}$$

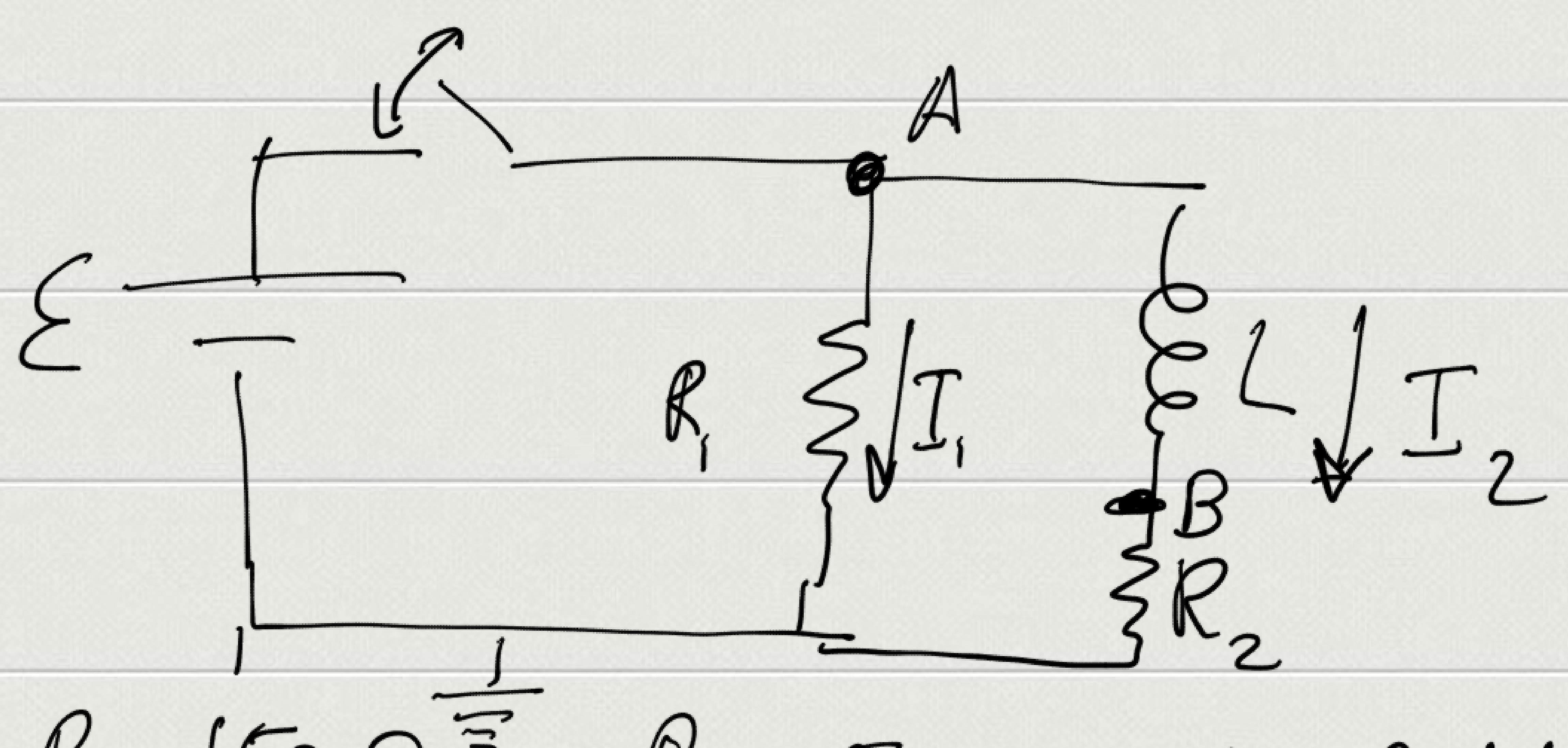
If instead we connect the coils like in figure (c) $I_1 = -I_2$. Write $I = I_1$. Then emf $= \mathcal{E}_1 - \mathcal{E}_2$. Subtract off equations (1) and (2). This gives

$$\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2 = -L_1\frac{dI}{dt} - M\frac{d(-I)}{dt} + L_2\frac{d(-I)}{dt} + M\frac{dI}{dt}$$

$$\mathcal{E} = -(L_1 + L_2 - 2M)\frac{dI}{dt} \qquad \boxed{L'' = L_1 + L_2 - 2M}$$

(c) Since $L'' \geq 0$, must have $\boxed{M \leq \frac{L_1 + L_2}{2}}$

7.41



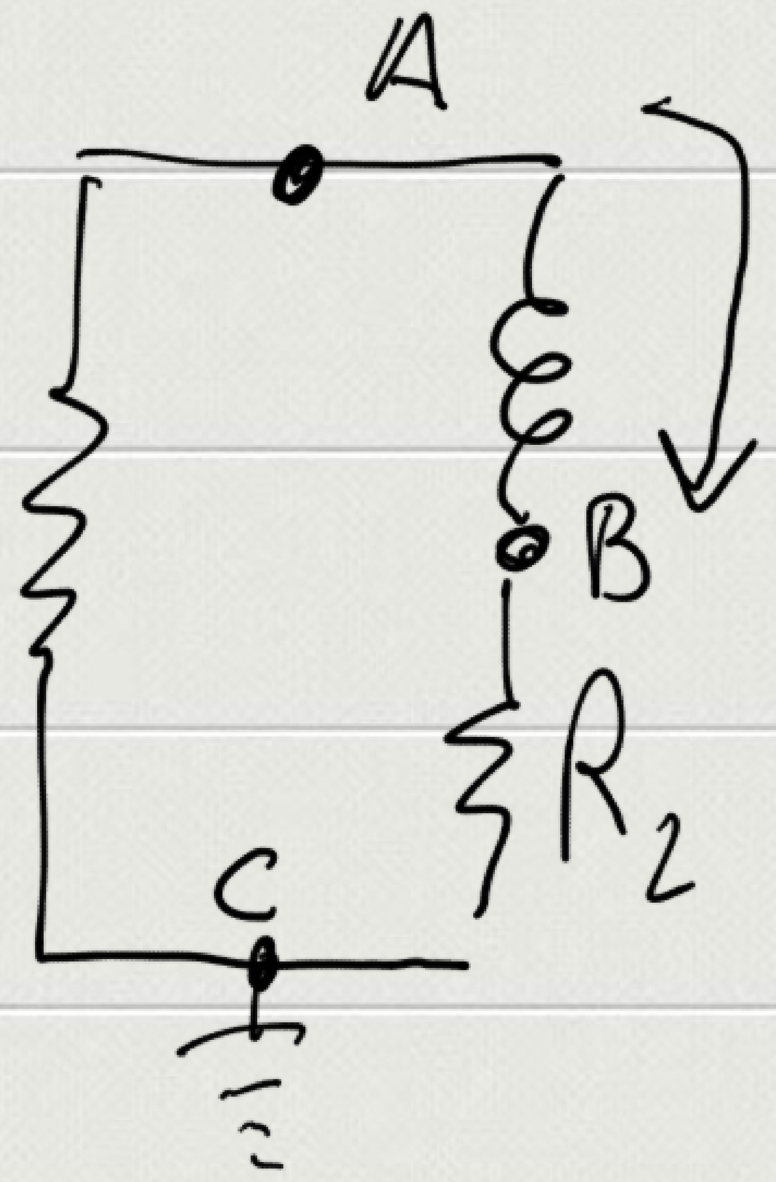
$\mathcal{E} = 10V$ $R_1 = 150\,\Omega$ $R_2 = 50\,\Omega$ $L = 0.1\,H$

When the switch has been closed for a long time, all transients have died out. Thus $V_A = \mathcal{E} = 10\,V$ $\qquad \underline{\underline{V_B = V_A = 10\,V}}$

$$I_1 = V_A / R_1 = 0.067\,A$$

$$I_2 = \frac{V_B}{R_2} = \frac{V_A}{R_2} = 0.20\,A$$

When we open the switch, the circuit now looks like this



$V_A - V_B = L\,\dfrac{dI}{dt}$ (1)

$V_B - V_C = V_B = I R_2$ (2)

$V_C - V_A = -V_A = I R_1$ (3)

$\underline{V_C = 0}$

$$L\frac{dI}{dt} + I(R_1+R_2) = 0$$

Assuming that the switch is closed at $\underline{t=0}$, the solution is $I = I_0 e^{-t/\tau}$

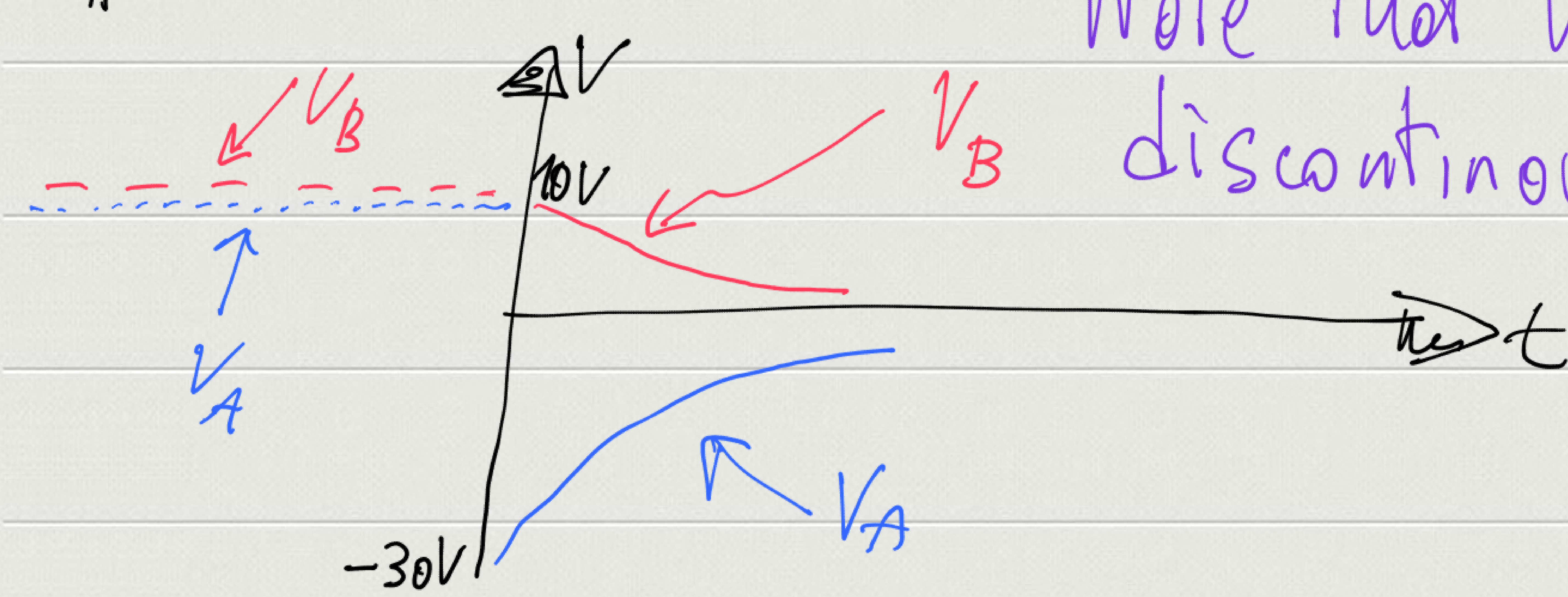
with $\tau = L/R_1+R_2 = 0.1/200$ sec $= \underline{0.5\text{ msec}}$

I_0 is given by the value from the previous page ie $I_0 = 0.2A$ _ Then from eqtns 2 & 3 we can get V_A and V_B as a function of time.

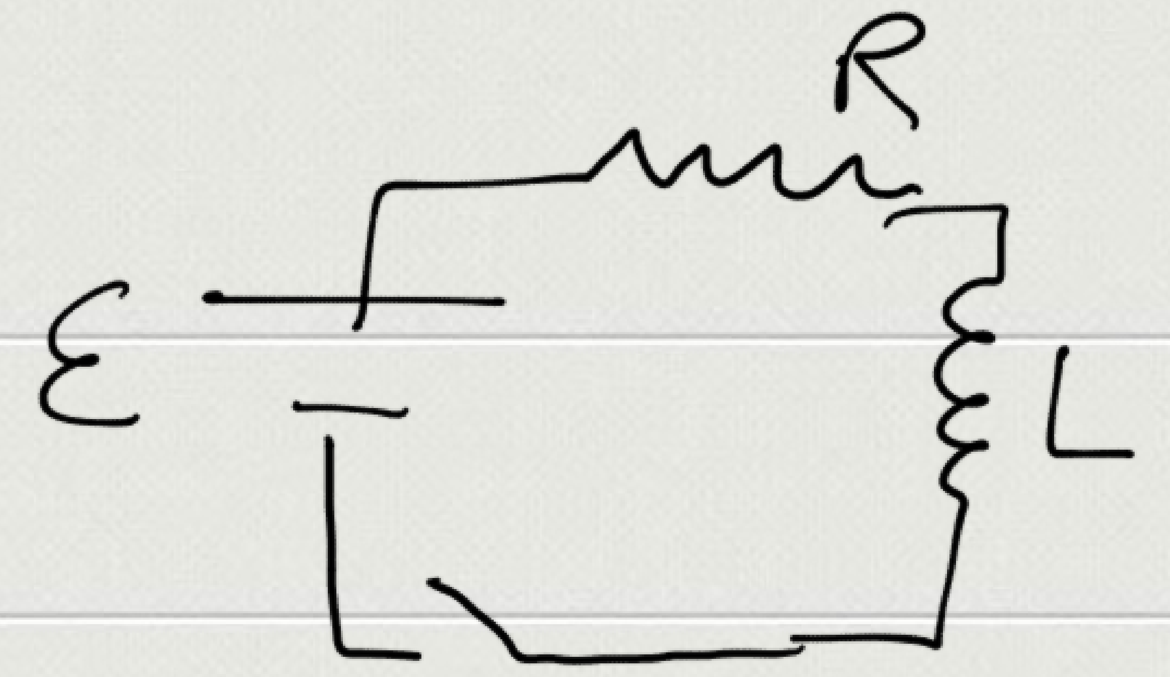
$V_B = I R_2 = 10V e^{-t/\tau}$

$V_A = -I R_1 = -30V e^{-t/\tau}$

Note that V_A is discontinous !!



7.42



$\mathcal{E} = 12V$
$R = 0{,}01\Omega$
$L = 5\,10^{-4}H$

$$I = I_0\left(1 - e^{-t/\tau}\right) \qquad \tau = L/R \qquad I_0 = \mathcal{E}/R$$

Want $e^{-t/\tau} = 0.1$ $\quad \frac{t}{\tau} = \log 10$

$$t = \log 10 \cdot \frac{L}{R} = \underline{\underline{0.115\ \mathrm{sec}}}$$

$$E = \frac{1}{2} L I^2 = \frac{1}{2} L \left(\frac{9}{10} I_0\right)^2 = \frac{81}{200} L \frac{\mathcal{E}^2}{R^2}$$

$$E = \frac{81}{200}\, 5\, 10^{-4}\, \frac{12^2}{10^{-4}}\ J = \underline{\underline{292\ J}}$$

For the energy delivered by the battery, we cannot simply answer "292 J" because there have been I^2R losses in the resistor. (Remember: I^2R is power, ie, dE/dt). We can do this two ways. (a) integrate I^2R and add it to 292 J; or (b) integrate $\mathcal{E} I$, which is the power instantaneously delivered by the battery. Let's do it both ways

$$E = \int_0^{t_1} \mathcal{E}_0 I \, dt \qquad t_1 = \log 10 \frac{L}{R}$$ ⑥

$$E = \mathcal{E}_0 I_0 \int_0^{t_1} (1 - e^{-t/\tau}) dt = \mathcal{E}_0 I_0 \left[t_1 + \tau e^{-t_1/\tau} - \tau \right]$$

Note: $e^{-t_1/\tau} = \frac{1}{10}$

$$E = \mathcal{E}_0 I_0 \left[t_1 - \frac{9}{10} \tau \right] = \frac{\mathcal{E}_0^2 L}{R^2} \left[\log 10 - 9/10 \right]$$

$$E = \frac{12^2 \, 5 \, 10^{-4}}{10^{-4}} \left[\log 10 - 9/10 \right] = \underline{\underline{1010 \, J}}$$

This means that $1010 \, J - 292 \, J = \underline{718 \, J}$ must have been dissipated in the resistor. Let's verify that

$$E_{Resistor} = \int I^2 R \, dt = I_0^2 R \int_0^{t_1} (1 - e^{-t/\tau})^2 dt$$

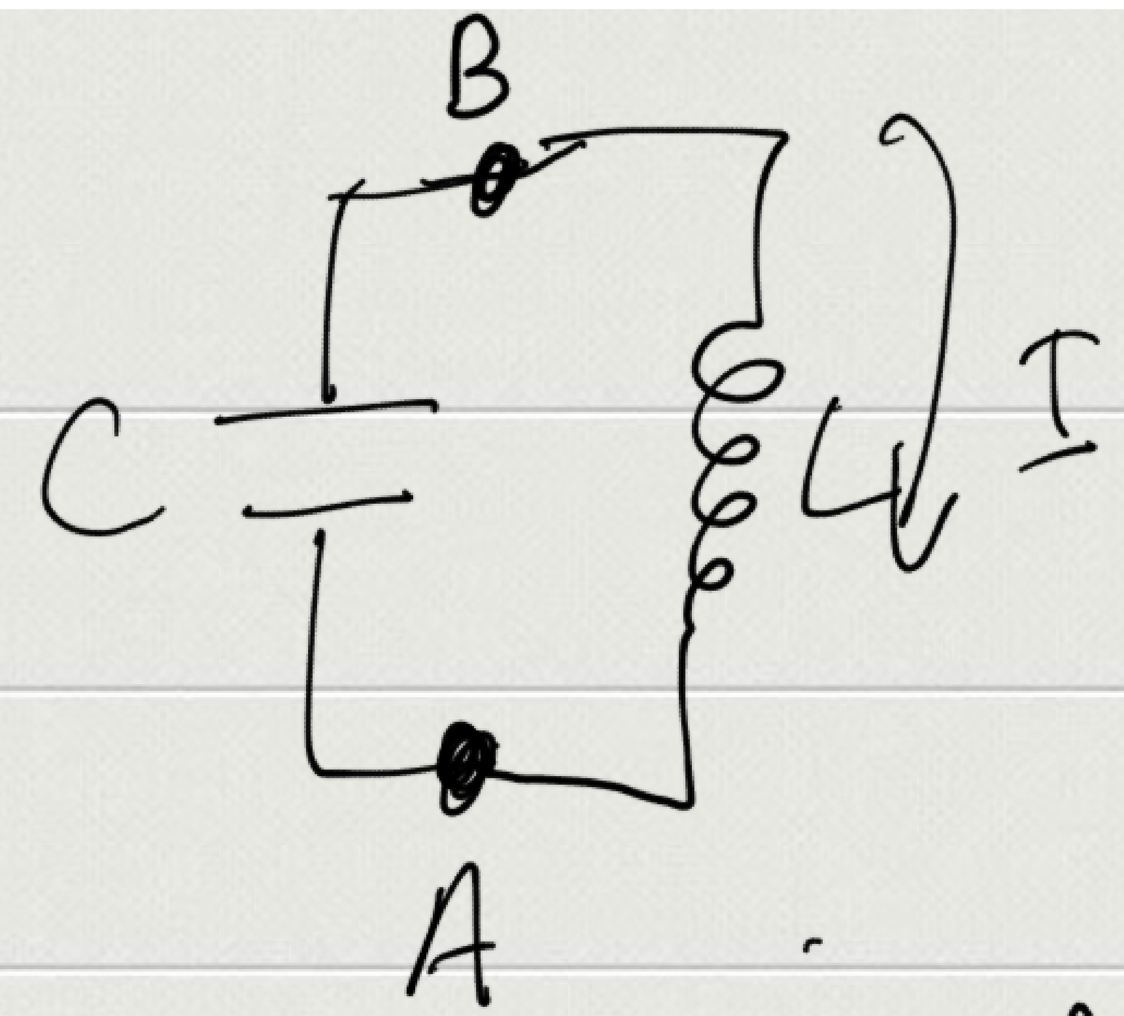
$$= \frac{\mathcal{E}_0^2}{R} \int_0^{t_1} (1 - 2 e^{-t/\tau} + e^{-2t/\tau}) dt =$$

$$= \frac{\mathcal{E}_0^2}{R} \left[t_1 + 2\tau e^{-t_1/\tau} - \tau - \frac{\tau}{2} e^{-2t_1/\tau} + \frac{\tau}{2} \right] \qquad = \frac{1}{100}$$

$$= \frac{\mathcal{E}_0^2}{R} \left[t_1 - 2 \cdot \frac{9}{10} \tau + \frac{1}{2} \frac{99}{100} \tau \right] = \frac{\mathcal{E}_0^2 L}{R^2} \left[\log 10 - \frac{18}{10} + \frac{99}{200} \right]$$

$$= \frac{12^2 \cdot 5 \cdot 10^{-4}}{10^{-4}} \left[0.997 \right] = \underline{\underline{718 \, J}}$$ ✓ **Yes!!**

8.16



$V_A - V_B = V_0 \cos\omega t$ (voltage across C)

$V_B - V_A = -V_0 \cos\omega t$ (voltage across L)

Current $L \frac{dI}{dt} = -V_0 \cos\omega t \Rightarrow I = -\frac{V_0}{\omega L} \sin\omega t$

or $I = V_0 \sqrt{\frac{C}{L}} \sin\omega t$

Energy in the inductor $U_L = \frac{1}{2} L I^2 = \frac{1}{2} V_0^2 C \sin^2\omega t$

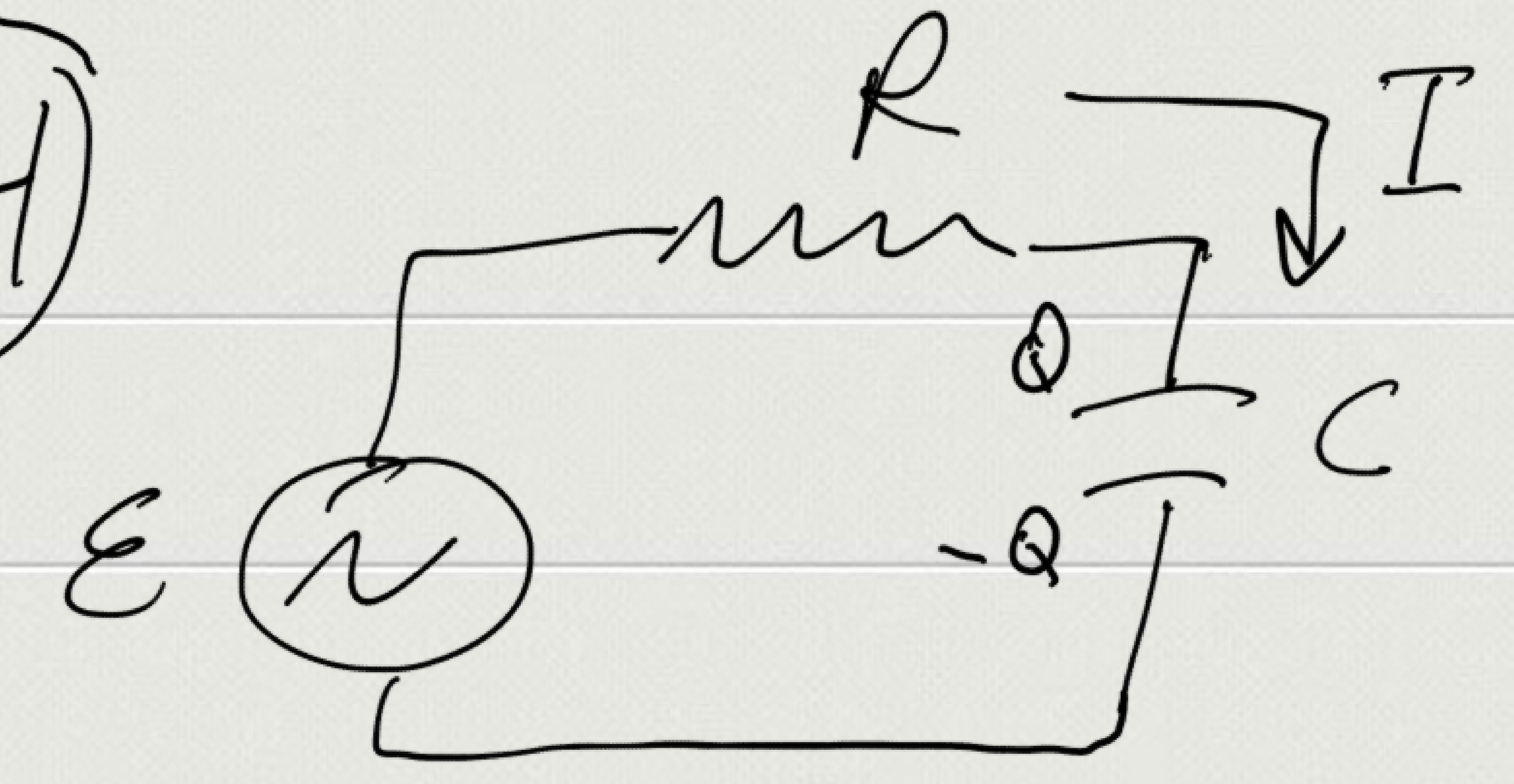
Energy in the capacitor $U_C = \frac{1}{2} C (V_A - V_B)^2 = \frac{1}{2} V_0^2 C \cos^2\omega t$

Note: $U_L + U_C = \frac{1}{2} C V_0^2$ is constant

At $t = 0$ $\quad U_L = 0 \quad U_C = \frac{1}{2} C V_0^2$

At $t = \frac{\pi}{2\omega}$ $\quad U_L = \frac{1}{2} C V_0^2 \quad U_C = 0$

8.24



$$IR + \frac{Q}{C} = \mathcal{E}_0 \cos\omega t$$

$$R\frac{dQ}{dt} + \frac{Q}{C} = \mathcal{E}_0 \cos\omega t = \mathcal{E}_0 e^{i\omega t}$$

Write $Q = \tilde{Q}_0 e^{i\omega t} \Rightarrow \frac{dQ}{dt} = i\omega\tilde{Q}_0 e^{i\omega t}$

$$i\omega R\tilde{Q}_0 + \frac{\tilde{Q}_0}{C} = \mathcal{E}_0 \qquad \tilde{Q}_0 = \frac{C\mathcal{E}_0}{1 + i\omega Rc}$$

But we wanted current, so $I = \tilde{I}_0 e^{i\omega t}$

and $\tilde{I}_0 = i\omega\tilde{Q}_0 = \frac{i\omega C\mathcal{E}_0}{1 + i\omega RC} = \frac{i\omega C\mathcal{E}_0(1 - i\omega RC)}{1 + \omega^2R^2C^2}$

$$\tilde{I}_0 = \frac{\mathcal{E}_0\omega^2RC^2 + i\mathcal{E}_0\omega C}{1 + \omega^2R^2C^2}$$

Now write $\tilde{I}_0 = I_0 e^{i\phi}$

$$\tan\phi = \frac{\mathcal{E}_0\omega C}{\mathcal{E}_0\omega^2RC^2} = \frac{1}{\omega RC}$$

$$I_0 = \frac{\mathcal{E}_0\sqrt{\omega^4R^2C^4 + \omega^2C^2}}{1 + \omega^2R^2C^2} = \frac{\mathcal{E}_0\omega C\sqrt{1 + \omega^2R^2C^2}}{1 + \omega^2R^2C^2}$$

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$$I_0 = \frac{\omega C \mathcal{E}_0}{\sqrt{1+\omega^2 R^2 C^2}} = \frac{\mathcal{E}_0}{\sqrt{R^2 + 1/\omega^2 C^2}}$$

Now take the real part of $\tilde{I} e^{i\omega t} = I_0 e^{i(\omega t + \phi)}$

$$I(t) = \frac{\mathcal{E}_0}{\sqrt{R^2 + 1/\omega^2 C^2}} \cos(\omega t + \phi)$$

with $\phi = \tan^{-1} \frac{1}{R\omega C}$

For large ω, amplitude $\Rightarrow \mathcal{E}_0/R$

phase $\Rightarrow 0$

This makes sense, because at high ω the capacitor is like a short

For small ω, the amplitude goes to zero. This makes sense because for DC currents the capacitor does not conduct