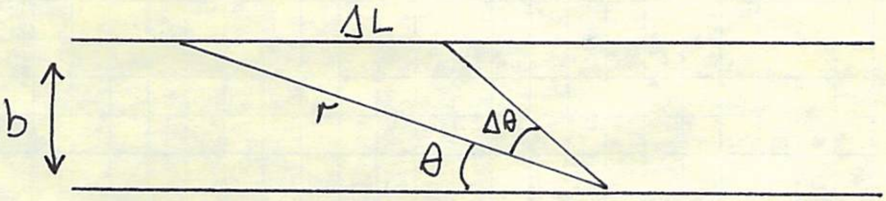
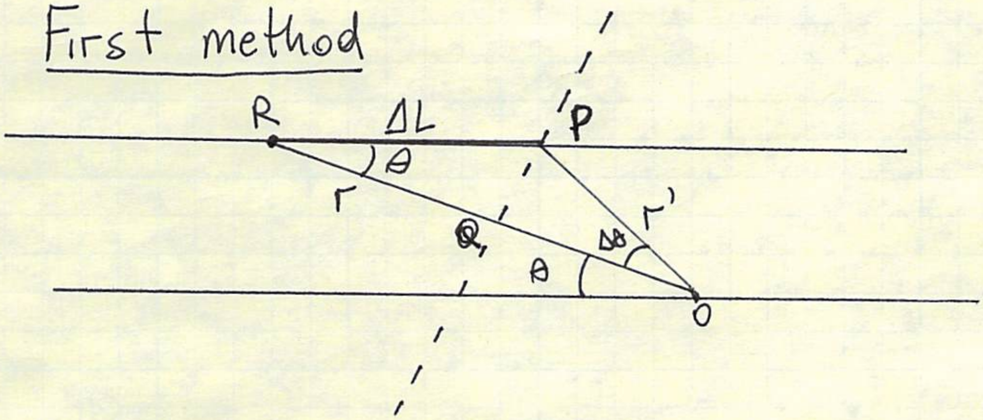


We want to show that as $\Delta\theta \to 0$, $\Delta L \to \dfrac{r\Delta\theta}{\sin\theta}$

<u>First method</u>



The dashed line is drawn such that it is $\perp$ to $\overrightarrow{OR}$ and such that $|\overrightarrow{OQ}| = |\overrightarrow{OP}| = r'$

Imagine a circle of radius r' centered around O.

The length of the arc of circle $\widehat{PQ} = r'\Delta\theta$

As $\Delta\theta \to 0$ $\quad \widehat{PQ} \approx |\overrightarrow{PQ}|$ and $r' \approx r$

$\Rightarrow \quad |\overrightarrow{PQ}| \approx r'\Delta\theta \approx r\Delta\theta$

From the PQR triangle

$$|\overrightarrow{PQ}| = |\overrightarrow{PR}| \sin\theta$$

$$|\overrightarrow{PQ}| = \Delta L \sin\theta$$

$$\Delta L = \frac{|\overrightarrow{PQ}|}{\sin\theta} \approx \frac{r\Delta\theta}{\sin\theta}$$

Second Method

From the PQR triangle $|\vec{PQ}| = \Delta L \sin\theta$

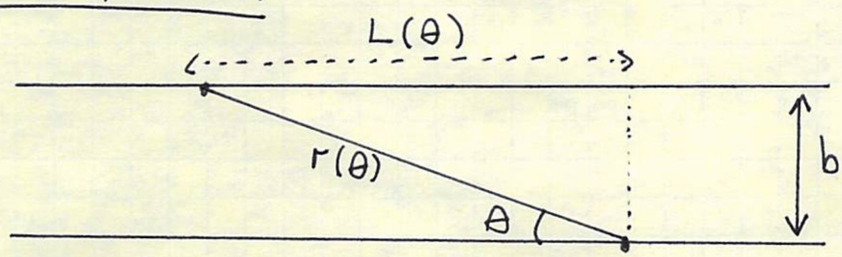
From the PQO triangle $|\vec{PQ}| = |\vec{PO}| \sin\Delta\theta$

$$\Rightarrow \Delta L = \frac{|\vec{PO}| \sin\Delta\theta}{\sin\theta} = \frac{r' \sin\Delta\theta}{\sin\theta}$$

But if $\Delta\theta \to 0$ $r' \to r$ and $\sin\Delta\theta \to \Delta\theta$

$$\Rightarrow \Delta L \approx \frac{r\Delta\theta}{\sin\theta}$$

Third method



$$\cos\theta \, r(\theta) = L(\theta)$$

But $r(\theta) = \dfrac{b}{\sin\theta}$

$$\Rightarrow L(\theta) = b \cot\theta$$

$$\frac{dL}{d\theta} = b \frac{d \cot\theta}{d\theta} = -\frac{b}{\sin^2\theta}$$

$$\Delta L \approx \left|\frac{dL}{d\theta}\right| \Delta\theta = \frac{b}{\sin^2\theta} \Delta\theta = \frac{r}{\sin\theta} \Delta\theta$$